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# A Lot of Examples of Generalized Weak Bi-Frobenius Algebras

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**Abstract:** In this paper, by considering the tensor product of a bi-Frobenius algeba and a weak Hopf algebra, a lot of examples of the generalized weak bi-Frobenius algebras are given, such as the 16-dimensional, 24-dimensional and 40-dimensional GWBF algebras. They provide a common generalization of weak Hopf algebras introduced by Böhm, Nill, Szlachányi, and of bi-Frobenius algebras introduced by Doi and Takeuchi.

**Keywords:** Examples, Bi-Frobenius Algebras, Generalized Weak Bi-Frobenius Algebras

## 1. Introduction

Doi and Takeuchi axiomatized these above facts to obtain the notion of bi-Frobenius algebras in [1-2], which generalizes the notion of finite dimensional Hopf algebras. A bi-Frobenius algebra is a Frobenius algebra and a Frobenius coalgebra with a pair of right integrals $(\varphi,\ t)$ and an antipode- like anti-automorphism $S$. Further research on bi-Frobenius algebras could be found in [3-7].

In 2013, Chen and Wang [3] introduced the notion of a generalized weak bi-Frobenius algebra, which is a generalization of weak Hopf algebras [8-10] and bi-Frobenius algebras [2]. They investigated some properties of various modular elements of generalized weak bi-Frobenius algebras [11-13]. They also provided some examples of generalized weak bi-Frobenius algebras. However, their examples are in fact face algebras or the direct products of a weak Hopf algebra and a bi-Frobenius algebra [13-15]. So there still lacks non-trivial examples. This is the motivation of this manuscript.

The general organization of this paper is as the following. In Section 1, some notions of generalized weak bi-Frobenius algebras are recalled. In Section 2, some non-trivial examples of generalized weak bi-Frobenius algebras are given.

## 2. Preliminaries

In this section the basic definitions related to the paper are recalled.

Let $(H,\ m,\ \eta)$ be a finite dimensional algebra with unit 1 and coalgebra with counit $\varepsilon,\ 0 \neq \psi \in H^*$. There is a bijective map $S: H \to H$ satisfying, for all $h,\ g \in H$,

$$\psi\ (hg_1)\ S\ (g_2) = \psi\ (h_1 g)\ h_2$$

The data $(H,\ \psi,\ S)$ is called a generalized weak biFrobenius algebra (or GWBF-algebra) if the following conditions hold:

Unit 1 satisfies the following identities:

$$(\Delta \otimes id)\Delta\ (1) = (\Delta\ (1) \otimes 1)(1 \otimes \Delta\ (1)) = (1 \otimes \Delta\ (1))(\Delta\ (1) \otimes 1);$$

Counit $\varepsilon$ satisfies the following identities:

$$\varepsilon\ (fgh) = \varepsilon\ (fg_1)\ \varepsilon\ (g_2 h) = \varepsilon\ (fg_2)\ \varepsilon\ (g_1 h);$$

The pair $(H,\ \psi)$ is a Frobenius algebra, i.e.,

$$\psi' H = H^*;$$

The map $S$ is an anti-algebra map, i.e.,

$$S(hg) = S(g)S(h),\ S(1) = 1;$$

The map $S$ is an anti-algebra map, i.e.,

$$\Delta\left(S(h)\right) = \sum S(h_2) \otimes S(h_1),\ \varepsilon\left(S(h)\right) = \varepsilon(h).$$

For all $f,\ g,\ h \in H$.

Let $(H, \psi, S)$ be a GWBF-algebra. By a *GWBF-subalgebra* it means that a subalgebra and subcoalgebra $K$ with $S(K) \subset K$ and $(K, \psi', S|_K)$ becomes a GWBF-algebra for some $\psi' \in K^*$.

# 3. The Examples of GWBF Algebra

In this section some examples of generalized weak biFrobenius algebra are given.

### *3.1. The 16-Dimensional GWBF Algebra*

Define $A$ as the 16-dimensional associative algebra generated by a set $\{X_{ij} | i = 0,\ 1,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) $X_{00}$ is the unit;

(2) $X_{ij}X_{st} = X_{pq}$, where $p = 0$ if $i = 0$ and $s = 0$, $p = 1$ if $i = 1$ or $s = 1$; and $q = (j + t)$ mod8.

(3) The coproduct and counit on $A$ are given by:

$$\Delta(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + X_{10} \otimes X_{10};$$

$$\Delta(X_{10}) = X_{10} \otimes X_{10};$$

$$\Delta(X_{01}) = (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + X_{11} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + X_{10} \otimes X_{11};$$

$$\Delta(X_{11}) = X_{11} \otimes X_{10} + X_{10} \otimes X_{11};$$

$$\Delta(X_{02}) = (X_{02} - X_{12}) \otimes (X_{00} - X_{10}) + X_{12} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + X_{10} \otimes X_{12};$$

$$\Delta(X_{12}) = X_{12} \otimes X_{10} + X_{10} \otimes X_{12};$$

$$\Delta(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + X_{10} \otimes X_{13} + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + X_{11} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + X_{12} \otimes X_{11} + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + X_{13} \otimes X_{10};$$

$$\Delta(X_{13}) = X_{10} \otimes X_{13} + X_{11} \otimes X_{12} + X_{12} \otimes X_{11} + X_{13} \otimes X_{10};$$

$$\Delta(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + X_{10} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{00} - X_{10}) + X_{14} \otimes X_{10};$$

$$\Delta(X_{14}) = X_{10} \otimes X_{14} + X_{14} \otimes X_{10};$$

$$\Delta(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + X_{10} \otimes X_{15} + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + X_{11} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + X_{14} \otimes X_{11} + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + X_{15} \otimes X_{10};$$

$$\Delta(X_{15}) = X_{10} \otimes X_{15} + X_{11} \otimes X_{14} + X_{14} \otimes X_{11} + X_{15} \otimes X_{10};$$

$$\Delta(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + X_{10} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + X_{12} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + X_{14} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + X_{16} \otimes X_{10};$$

$$\Delta(X_{16}) = X_{10} \otimes X_{16} + X_{12} \otimes X_{14} + X_{14} \otimes X_{12} + X_{16} \otimes X_{10};$$

$$\Delta(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + X_{10} \otimes X_{17} + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + X_{11} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + X_{12} \otimes X_{15} + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + X_{13} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + X_{14} \otimes X_{13} + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + X_{15} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + X_{16} \otimes X_{11} + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + X_{17} \otimes X_{10};$$

$$\Delta(X_{17}) = X_{10} \otimes X_{17} + X_{11} \otimes X_{16} + X_{12} \otimes X_{15} + X_{13} \otimes X_{14} + X_{14} \otimes X_{13} + X_{15} \otimes X_{12} + X_{16} \otimes X_{11} + X_{17} \otimes X_{10};$$

$$\varepsilon\ (X_{00}) = 2, \varepsilon\ (X_{10}) = 1,\ \varepsilon\ (X_{ij}) = 0 \text{ (for } j \geq 1);$$

(4) $\psi\ (X_{07}) = \frac{3}{2},\ \psi l\ (X_{17}) = \frac{1}{2},\ \psi(X_{ij}) = 0$ (for $j \leq 6$);

(5) $S = id_A$;

Then $(A,\ m,\ \eta,\ \Delta,\ \varepsilon,\ \psi,\ S)$ is a GWBF-algebra.

Let $K$ be the subspace generated by $X_{00},\ X_{10},\ X_{02},\ X_{12},\ X_{04},\ X_{14},\ X_{06},\ X_{16}$. It is a subalgebra and subcoalgebra of $A$. Define $\phi \in K^*$ by $\phi(X_{06}) = \frac{3}{2},\ \phi(X_{16}) = \frac{1}{2},\ \phi(X_{ij}) = 0$(for $j = 0,\ 2,\ 4$). Then $(K,\ \phi,\ S|_K)$ is a GWBF-subalgebra of $A$.

### *3.2. The 24-Dimensional GWBF Algebra (Type 1)*

Define $B$ as the 24-dimensional associative algebra generated by a set $\{X_{ij}|i = 0,\ 1,\ 2,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) $X_{00}$ is the unit;

(2) $X_{ij}X_{st} = X_{pq}$, where $p = 0$ if $i = 0$ and $s = 0$, $p = 2$ if $i = 2$ or $s = 2$, $p = 1$ for the other condition; and $q = (j + t)$ mod8.

(3) The coproduct and counit on $A$ are given by:

$$\Delta(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + X_{10} \otimes X_{10};$$

$$\Delta(X_{10}) = X_{10} \otimes X_{10};$$

$$\Delta(X_{20}) = (X_{10} - X_{20}) \otimes X_{20} + X_{20} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{01}) = (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + X_{11} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + X_{10} \otimes X_{11};$$

$$\Delta(X_{11}) = X_{11} \otimes X_{10} + X_{10} \otimes X_{11};$$

$$\Delta(X_{21}) = (X_{10} - X_{20}) \otimes X_{21} + (X_{11} - X_{21}) \otimes X_{20} + X_{20} \otimes (X_{11} - X_{21}) + X_{21} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{02}) = (X_{02} - X_{12}) \otimes (X_{00} - X_{10}) + X_{12} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + X_{10} \otimes X_{12};$$

$$\Delta(X_{12}) = X_{12} \otimes X_{10} + X_{10} \otimes X_{12};$$

$$\Delta(X_{22}) = (X_{10} - X_{20}) \otimes X_{22} + (X_{12} - X_{22}) \otimes X_{20} + X_{20} \otimes (X_{12} - X_{22}) + X_{22} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + X_{10} \otimes X_{13} + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + X_{11} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + X_{12} \otimes X_{11} + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + X_{13} \otimes X_{10};$$

$$\Delta(X_{13}) = X_{10} \otimes X_{13} + X_{11} \otimes X_{12} + X_{12} \otimes X_{11} + X_{13} \otimes X_{10};$$

$$\Delta(X_{23}) = (X_{10} - X_{20}) \otimes X_{23} + X_{20} \otimes (X_{13} - X_{23}) + (X_{11} - X_{21}) \otimes X_{22} + X_{21} \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes X_{21} + X_{22} \otimes (X_{11} - X_{21}) + (X_{13} - X_{23}) \otimes X_{20} + X_{23} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + X_{10} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{00} - X_{10}) + X_{14} \otimes X_{10};$$

$$\Delta(X_{14}) = X_{10} \otimes X_{14} + X_{14} \otimes X_{10};$$

$$\Delta(X_{24}) = (X_{10} - X_{20}) \otimes X_{24} + (X_{14} - X_{24}) \otimes X_{20} + X_{20} \otimes (X_{14} - X_{24}) + X_{24} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + X_{10} \otimes X_{15} + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + X_{11} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + X_{14} \otimes X_{11} + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + X_{15} \otimes X_{10};$$

$$\Delta(X_{15}) = X_{10} \otimes X_{15} + X_{11} \otimes X_{14} + X_{14} \otimes X_{11} + X_{15} \otimes X_{10};$$

$$\Delta(X_{25}) = (X_{10} - X_{20}) \otimes X_{25} + X_{20} \otimes (X_{15} - X_{25}) + (X_{11} - X_{21}) \otimes X_{24} + X_{21} \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes X_{21} + X_{24} \otimes (X_{11} - X_{21}) + (X_{15} - X_{25}) \otimes X_{20} + X_{25} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + X_{10} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + X_{12} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + X_{14} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + X_{16} \otimes X_{10};$$

$$\Delta(X_{16}) = X_{10} \otimes X_{16} + X_{12} \otimes X_{14} + X_{14} \otimes X_{12} + X_{16} \otimes X_{10};$$

$$\Delta(X_{26}) = (X_{10} - X_{20}) \otimes X_{26} + X_{20} \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes X_{24} + X_{22} \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes X_{22} + X_{24} \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes X_{20} + X_{26} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + X_{10} \otimes X_{17} + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + X_{11} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + X_{12} \otimes X_{15} + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + X_{13} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + X_{14} \otimes X_{13} + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + X_{15} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + X_{16} \otimes X_{11} + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + X_{17} \otimes X_{10};$$

$$\Delta(X_{17}) = X_{10} \otimes X_{17} + X_{11} \otimes X_{16} + X_{12} \otimes X_{15} + X_{13} \otimes X_{14} + X_{14} \otimes X_{13} + X_{15} \otimes X_{12} + X_{16} \otimes X_{11} + X_{17} \otimes X_{10};$$

$$\Delta(X_{27}) = (X_{10} - X_{20}) \otimes X_{27} + X_{20} \otimes (X_{17} - X_{27}) + (X_{11} - X_{21}) \otimes X_{26} + X_{21} \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes X_{25} + X_{22} \otimes (X_{15} - X_{25}) + (X_{13} - X_{23}) \otimes X_{24} + X_{23} \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes X_{23} + X_{24} \otimes (X_{13} - X_{23}) + (X_{15} - X_{25}) \otimes X_{22} + X_{25} \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes X_{21} + X_{26} \otimes (X_{11} - X_{21}) + (X_{17} - X_{27}) \otimes X_{20} + X_{27} \otimes (X_{10} - X_{20});$$

$$\varepsilon(X_{00}) = 2,\ \varepsilon(X_{10}) = 1,\ \varepsilon(X_{ij}) = 0(\text{for } i = 2 \text{ or } j \geq 1);$$

(4) $\psi(X_{07}) = 3,\ \psi(X_{17}) = 2,\ \psi(X_{27}) = 1,\ \psi(X_{ij}) = 0(\text{for } j \leq 6);$

(5) $S = id_B,$

Then $(B,\ m,\ \eta,\ \Delta,\ \varepsilon,\ \psi,\ S)$ is a GWBF-algebra. Let $L$ be the subspace generated by $X_{00},\ X_{10},\ X_{20},\ X_{02},\ X_{12},\ X_{22},\ X_{04},\ X_{14},\ X_{24},\ X_{06},\ X_{16}$ and $X_{26}$. It is a subalgebra and subcoalgebra of $B$.

Moreover, define $\rho \in L^*$ by

$$\rho(X_{06}) = 3,\ \rho(X_{16}) = 2,\ \rho(X_{26}) = 1,\ \rho(X_{ij}) = 0 \text{ (for } j = 0,\ 2,\ 4),$$

Then $(L,\ \rho,\ S|_L)$ is a GWBF-subalgebra of $B$.

### *3.3. The 24-Dimensional GWBF Algebra (Type 2)*

Define $B'$ as the 24-dimensional associative algebra generated by a set $\{X_{ij} | i = 0,\ 1,\ 2,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) $X_{00}$ is the unit;

(2) $X_{ij}X_{st} = X_{pq}$, where $p = 0$ if $i = 0$ and $s = 0$, $p = 2$ if $i = 2$ or $s = 2$, $p = 1$ for the other condition; and $q = (j + t)$ mod8.

(3) The coproduct and counit on $A$ are given by:

$$\Delta(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{20};$$

$$\Delta(X_{10}) = (X_{10} - X_{20}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{20};$$

$$\Delta(X_{20}) = X_{20} \otimes X_{20};$$

$$\Delta(X_{01}) = (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{11} - X_{21}) + (X_{11} - X_{21}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta(X_{11}) = (X_{10} - X_{20}) \otimes (X_{11} - X_{21}) + (X_{11} - X_{21}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta(X_{21}) = X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta(X_{02}) = (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + (X_{02} - X_{12}) \otimes (X_{00} - X_{10})) + (X_{10} - X_{20}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{10} - X_{20}) + X_{22} \otimes X_{20} + X_{20} \otimes X_{22};$$

$$\Delta(X_{12}) = (X_{10} - X_{20}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{10} - X_{20}) + X_{22} \otimes X_{20} + X_{20} \otimes X_{22};$$

$$\Delta(X_{22}) = X_{22} \otimes X_{20} + X_{20} \otimes X_{22};$$

$$\Delta(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{13} - X_{23}) + (X_{11} - X_{21}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{11} - X_{21}) + (X_{13} - X_{23}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta(X_{13}) = (X_{10} - X_{20}) \otimes (X_{13} - X_{23}) + (X_{11} - X_{21}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{11} - X_{21}) + (X_{13} - X_{23}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta(X_{23}) = X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{00} - X_{10})) + (X_{10} - X_{20}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{10} - X_{20}) + X_{24} \otimes X_{20} + X_{20} \otimes X_{24};$$

$$\Delta(X_{14}) = (X_{10} - X_{20}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{10} - X_{20}) + X_{24} \otimes X_{20} + X_{20} \otimes X_{24};$$

$$\Delta(X_{24}) = X_{24} \otimes X_{20} + X_{20} \otimes X_{24};$$

$$\Delta(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{15} - X_{25}) + (X_{11} - X_{21}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{11} - X_{21}) + (X_{15} - X_{25}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta(X_{15}) = (X_{10} - X_{20}) \otimes (X_{15} - X_{25}) + (X_{11} - X_{21}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{11} - X_{21}) + (X_{15} - X_{25}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta(X_{25}) = X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{16}) = (X_{10} - X_{20}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{26}) = X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{17} - X_{27}) + (X_{11} - X_{21}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{15} - X_{25}) + (X_{13} - X_{23}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{13} - X_{23}) + (X_{15} - X_{25}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{11} - X_{21}) + (X_{17} - X_{27}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\Delta(X_{17}) = (X_{10} - X_{20}) \otimes (X_{17} - X_{27}) + (X_{11} - X_{21}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{15} - X_{25}) + (X_{13} - X_{23}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{13} - X_{23}) + (X_{15} - X_{25}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{11} - X_{21}) + (X_{17} - X_{27}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\Delta(X_{27}) = X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\varepsilon(X_{00}) = 3,\ \varepsilon(X_{10}) = 3,\ \varepsilon(X_{20}) = 1,\ \varepsilon(X_{ij}) = 0 \text{ (for } j \geq 1);$$

(4) $\iota(X_{07}) = 2,\ \iota(X_{17}) = 1,\ \iota(X_{27}) = \frac{1}{2},\ \iota(X_{ij}) = 0$ (for $j \leq 6$);

(5) $S = id_B$,

The $(B', m, \eta, \Delta, \varepsilon, \iota, S)$ is a GWBF-algebra. Let $L'$ be the subspace generated by $X_{00}$, $X_{10}$, $X_{20}$, $X_{02}$, $X_{12}$, $X_{22}$, $X_{04}$, $X_{14}$, $X_{24}$, $X_{06}$, $X_{16}$ and $X_{26}$. It is a subalgebra and subcoalgebra of $B'$.

Define $\rho \in L^*$ by

$$\rho(X_{06}) = 2,\ \rho(X_{16}) = 1,\ \rho(X_{26}) = \frac{1}{2},\ \rho(X_{ij}) = 0 \text{ (for } j = 0,\ 2,\ 4),$$

Then $(L',\ \rho,\ S|_{L'})$ is a GWBF-subalgebra of $B'$.

### *3.4. The 40-Dimensional GWBF Algebra*

Define $C$ as the 40-dimensional associative algebra generated by a set $\{X_{ij} | i = 0,\ 1,\ 2,\ 3,\ 4,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) $X_{00}$ is the unit;

(2) $X_{ij}X_{0t} = X_{0t}X_{ij} = X_{iq},\ X_{ij}X_{1t} = X_{1t}X_{ij} = X_{iq},\ X_{2j}X_{2t} = X_{1q},\ X_{2j}X_{3t} = -X_{4q},\ X_{3j}X_{2t} = X_{4q},\ X_{2j}X_{4t} = -X_{3q},\ X_{4j}X_{2t} = X_{3q},\ X_{3j}X_{3t} = X_{0q},\ X_{3j}X_{4t} = X_{4t}X_{3j} = X_{0q},\ X_{4j}X_{4t} = X_{0q},$ where $q = (j + t) \bmod 8$ for $i = 0,\ 1,\ 2,\ 3,\ 4$ and $j,\ t = 0,\ 1,\ \cdots,\ 7$.

(3) The coproduct and counit on $A$ are given by:

$$\Delta(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + X_{10} \otimes X_{10};$$

$$\Delta(X_{10}) = X_{10} \otimes X_{10};$$

$$\Delta(X_{20}) = X_{20} \otimes X_{20};$$

$$\Delta(X_{30}) = X_{20} \otimes X_{30} + X_{30} \otimes X_{20};$$

$$\Delta(X_{40}) = X_{10} \otimes X_{40} + X_{40} \otimes X_{20};$$

$$\Delta(X_{01}) = (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + X_{10} \otimes X_{11} + (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + X_{11} \otimes X_{10};$$

$$\Delta(X_{11}) = X_{10} \otimes X_{11} + X_{11} \otimes X_{10};$$

$$\Delta(X_{21}) = X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta(X_{31}) = X_{20} \otimes X_{31} + X_{21} \otimes X_{30} + X_{30} \otimes X_{11} + X_{31} \otimes X_{10};$$

$$\Delta(X_{41}) = X_{10} \otimes X_{41} + X_{40} \otimes X_{21} + X_{11} \otimes X_{40} + X_{41} \otimes X_{20};$$

$$\Delta(X_{02}) = (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + X_{10} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{00} - X_{10}) + X_{12} \otimes X_{10};$$

$$\Delta(X_{12}) = X_{10} \otimes X_{12} + X_{12} \otimes X_{10};$$

$$\Delta(X_{22}) = X_{20} \otimes X_{22} + X_{22} \otimes X_{20};$$

$$\Delta(X_{32}) = X_{20} \otimes X_{32} + X_{30} \otimes X_{12} + X_{22} \otimes X_{30} + X_{32} \otimes X_{10};$$

$$\Delta(X_{42}) = X_{10} \otimes X_{42} + X_{40} \otimes X_{22} + X_{12} \otimes X_{40} + X_{42} \otimes X_{20};$$

$$\Delta(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + X_{10} \otimes X_{13} + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + X_{11} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + X_{12} \otimes X_{11} + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + X_{13} \otimes X_{10};$$

$$\Delta(X_{13}) = X_{10} \otimes X_{13} + X_{11} \otimes X_{12} + X_{12} \otimes X_{11} + X_{13} \otimes X_{10};$$

$$\Delta(X_{23}) = X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta(X_{33}) = X_{20} \otimes X_{33} + X_{30} \otimes X_{13} + X_{21} \otimes X_{32} + X_{31} \otimes X_{12} + X_{22} \otimes X_{31} + X_{32} \otimes X_{11} + X_{23} \otimes X_{30} + X_{33} \otimes X_{10};$$

$$\Delta(X_{43}) = X_{10} \otimes X_{43} + X_{40} \otimes X_{23} + X_{11} \otimes X_{42} + X_{41} \otimes X_{22} + X_{12} \otimes X_{41} + X_{42} \otimes X_{21} + X_{13} \otimes X_{40} + X_{43} \otimes X_{20};$$

$$\Delta(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + X_{10} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{00} - X_{10}) + X_{14} \otimes X_{10};$$

$$\Delta(X_{14}) = X_{10} \otimes X_{14} + X_{14} \otimes X_{10};$$

$$\Delta(X_{24}) = X_{20} \otimes X_{24} + X_{24} \otimes X_{20};$$

$$\Delta(X_{34}) = X_{20} \otimes X_{34} + X_{30} \otimes X_{14} + X_{24} \otimes X_{30} + X_{34} \otimes X_{10};$$

$$\Delta(X_{44}) = X_{10} \otimes X_{44} + X_{40} \otimes X_{24} + X_{14} \otimes X_{40} + X_{44} \otimes X_{20};$$

$$\Delta(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + X_{10} \otimes X_{15} + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + X_{11} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + X_{14} \otimes X_{11} + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + X_{15} \otimes X_{10};$$

$$\Delta(X_{15}) = X_{10} \otimes X_{15} + X_{11} \otimes X_{14} + X_{14} \otimes X_{11} + X_{15} \otimes X_{10};$$

$$\Delta(X_{25}) = X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta(X_{35}) = X_{20} \otimes X_{35} + X_{30} \otimes X_{15} + X_{21} \otimes X_{34} + X_{31} \otimes X_{14} + X_{24} \otimes X_{31} + X_{34} \otimes X_{11} + X_{25} \otimes X_{30} + X_{35} \otimes X_{10};$$

$$\Delta(X_{45}) = X_{10} \otimes X_{45} + X_{40} \otimes X_{25} + X_{11} \otimes X_{44} + X_{41} \otimes X_{24} + X_{14} \otimes X_{41} + X_{44} \otimes X_{21} + X_{15} \otimes X_{40} + X_{45} \otimes X_{20};$$

$$\Delta(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + X_{10} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + X_{12} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + X_{14} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + X_{16} \otimes X_{10};$$

$$\Delta(X_{16}) = X_{10} \otimes X_{16} + X_{12} \otimes X_{14} + X_{14} \otimes X_{12} + X_{16} \otimes X_{10};$$

$$\Delta(X_{26}) = X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{36}) = X_{20} \otimes X_{36} + X_{30} \otimes X_{16} + X_{22} \otimes X_{34} + X_{32} \otimes X_{14} + X_{24} \otimes X_{32} + X_{34} \otimes X_{12} + X_{26} \otimes X_{30} + X_{36} \otimes X_{10};$$

$$\Delta(X_{46}) = X_{10} \otimes X_{46} + X_{40} \otimes X_{26} + X_{12} \otimes X_{44} + X_{42} \otimes X_{24} + X_{14} \otimes X_{42} + X_{44} \otimes X_{22} + X_{16} \otimes X_{40} + X_{46} \otimes X_{20};$$

$$\Delta(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + X_{10} \otimes X_{17} + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + X_{11} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + X_{12} \otimes X_{15} + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + X_{13} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + X_{14} \otimes X_{13} + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + X_{15} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + X_{16} \otimes X_{11} + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + X_{17} \otimes X_{10};$$

$$\Delta(X_{17}) = X_{10} \otimes X_{17} + X_{11} \otimes X_{16} + X_{12} \otimes X_{15} + X_{13} \otimes X_{14} + X_{14} \otimes X_{13} + X_{15} \otimes X_{12} + X_{16} \otimes X_{11} + X_{17} \otimes X_{10};$$

$$\Delta(X_{27}) = X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\Delta(X_{37}) = X_{20} \otimes X_{37} + X_{30} \otimes X_{17} + X_{21} \otimes X_{36} + X_{31} \otimes X_{16} + X_{22} \otimes X_{35} + X_{32} \otimes X_{15} + X_{23} \otimes X_{34} + X_{33} \otimes X_{14} + X_{24} \otimes X_{33} + X_{34} \otimes X_{13} + X_{25} \otimes X_{32} + X_{35} \otimes X_{12} + X_{26} \otimes X_{31} + X_{36} \otimes X_{11} + X_{27} \otimes X_{30} + X_{37} \otimes X_{10};$$

$$\Delta(X_{47}) = X_{10} \otimes X_{47} + X_{40} \otimes X_{27} + X_{11} \otimes X_{46} + X_{41} \otimes X_{26} + X_{12} \otimes X_{45} + X_{42} \otimes X_{25} + X_{13} \otimes X_{44} + X_{43} \otimes X_{24} + X_{14} \otimes X_{43} + X_{44} \otimes X_{23} + X_{15} \otimes X_{42} + X_{45} \otimes X_{22} + X_{16} \otimes X_{41} + X_{46} \otimes X_{21} + X_{17} \otimes X_{40} + X_{47} \otimes X_{20};$$

$$\varepsilon(X_{00}) = 2,\ \varepsilon(X_{10}) = 1,\ \varepsilon(X_{20}) = 1,\ \varepsilon(X_{ij}) = 0 \text{ (for } i = 3,\ 4 \text{ or } j \geq 1);$$

$$(4)\ \tau(X_{07}) = 1,\ \tau(X_{37}) = 1,\ \tau(X_{ij}) = 0 \text{ (for } i \neq 0,\ 3 \text{ or } j \leq 6);$$

(5) $S = id_C$.

Then $(C,\ m,\ \eta,\ \Delta,\ \varepsilon,\ \tau,\ S)$ is a GWBF-algebra. Let $M$ be the subspace generated by $X_{i0},\ X_{i2},\ X_{i4},\ X_{i6}$ for $i = 0,\ 1,\ 2,\ 3,\ 4$. It is a subalgebra and subcoalgebra of $C$. Define $\theta \in M^*$ by $\theta(X_{06}) = 1,\ \theta(X_{36}) = 1,\ \theta(X_{ij}) = 0$ (for $i \neq 0,\ 3$ or $j = 0,\ 2,\ 4$ ). Then $(M,\ \theta,\ S|_M)$ is a GWBF-subalgebra of $C$.

# 4. Conclusion

Poposition 4.1 Let $H$ and $K$ be two GWBF-algebras. Then the tensor product of $H$ and $K$ is also a GWBF-algebra.

Proof Deine the antipode as follows

$$S_{H\otimes \mathrm{K}} = S_H \otimes S_{\mathrm{K}},$$

And the integral as follows

$$\psi_{H\otimes \mathrm{K}} = \psi_H \otimes \psi_{\mathrm{K}},$$

It is obviously that $(H \otimes \mathrm{K},\ \mathrm{S}_{H\otimes \mathrm{K}},\ \psi_{H\otimes \mathrm{K}})$ is a GWBF-algebra under the tensor product algebra structure and tensor coproduct coalgebra structure.

Poposition 4.2 Let $H$ be a weak Hopf algebra, $K$ be a biFrobenius-algebra. Then the tensor product of $H$ and $K$ is a GWBF-algebra.

Proof The antipode is given by

$$S_{H\otimes \mathrm{K}} = S_H \otimes S_{\mathrm{K}},$$

And the integral is

$$\psi_{H\otimes \mathrm{K}} = \psi_H \otimes \psi_{\mathrm{K}},$$

Where $\psi_H$ is an integral of $H$.

Poposition 4.3 The GWBF-algebra $A$ introduced in Example 3.1 is a GWBF-subalgebra of $B$ in Example 3.2.

Proof Obviously $A$ is the 16-dimensional GWBF-subalgebra of B in Example 3.2 generated by $\{X_{ij} | i = 0,\ 1,\ j = 0,\ 1,\ \cdots,\ 7\}$.

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