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On the Intersection of a Hyperboloid and a Plane

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Abstract: The intersection topic is quite popular at an interdisciplinary level. It can be the friends of geometry, geodesy and others. The curves of intersection resulting in this case are not only ellipses but rather all types of conics: ellipses, hyperbolas and parabolas. In text books of mathematics usually only cases are treated, where the planes of intersection are parallel to the coordinate planes. Here the general case is illustrated with intersecting planes which are not necessarily parallel to the coordinate planes. We have developed an algorithm for intersection of a hyperboloid and a plane with a closed form solution. To do this, we rotate the hyperboloid and the plane until inclined plane moves parallel to the XY plane. In this situation, the intersection ellipse and its projection will be the same. This study aims to show how to obtain the center, the semi-axis and orientation of the intersection curve.

Keywords: Hyperboloid, Intersection, 3D Reverse Transformation, Plane

1. Introduction

1.1. Definition of Hyperboloid

Let a hyperboloid be given with the three positive semi axes a, b, c see Fig. 1

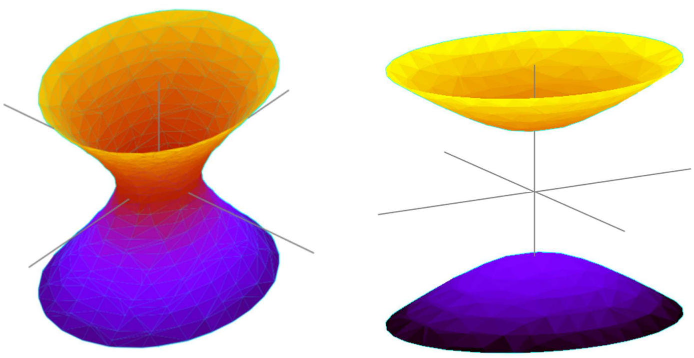
$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}-\frac{Z^2}{c^2}=\pm 1 \text{ (Hyperboloid equation)} \quad (1)$$

+1 where on the right hand side of (1) corresponds to a hyperboloid of one sheet, on the right hand side of -1 to a hyperboloid of two sheets.

1.2. Parameterization of Hyperboloid

Cartesian coordinates for the hyperboloids can be defined, similar to spherical coordinates, keeping the azimuth angle $\theta \in [0, 2\pi)$, but changing inclination v into hyperbolic trigonometric functions:

One-surface hyperboloid: $v \in (-\infty, \infty)$



a)Hyperboloid of one sheet b)Hyperboloid of two sheets

Figure 1. *Hyperboloid.*

1.3. Generalised Equation of Hyperboloid

More generally, an arbitrarily oriented hyperboloid, centered at **v**, is defined by the equation

$$(x\text{-}v)^T A\ (x\text{-}v) = 1$$

where *A* is a matrix and **x**, **v** are vectors.

The eigenvectors of *A* define the principal directions of the hyperboloid and the eigenvalues of A are the reciprocals of the squares of the semi-axes: $1/a^2$, $1/b^2$ and $1/c^2$. The one-sheet hyperboloid has two positive eigenvalues and one negative eigenvalue. The two-sheet hyperboloid has one positive eigenvalue and two negative eigenvalues. [5], [13], [4]

Basically everyone knows that intersection of a sphere and plane is a circle. But when we make a common solution of the sphere equation with the plane equation, it will give us an ellipse that is not a circle. While the solution in R3 space we have to eliminate one of the X,Y and Z parameters. In this case it is clear that there will be three possible solutions. Three of them are generally different ellipses from each other. For example, if we eliminate the parameter Z, we get the following ellipse equation on the XY plane that is the projection of the true intersection circle.

Similarly we find the intersection a Hyperboloid and a plane is an ellipse with a common solution. But it is not true curve. That is projection of the true intersection curve. The intersecting curve's plane is not parallel to the XY plane. This is why the two curves are different from each other.

In this study, we will be shown how to obtain the center, the semi-axis and orientation of the intersection curve.

As related to this subject limited number of studies was found in literature. Some of them are [9], [6] [10], [16], [15]. I think Klein's study [9] is a good study. But understanding his study requires familiarity with differential geometry. In this study we have put forward an alternative method addition to the Klein's study. We believe that our method is easier than to understand Klein's.

Our method is an easy way to understand the unfamiliar differential geometry. As also differently, we calculate the intersection curve's orientation information. Because the orientation information is extremely necessary especially in the curvature of surface.

Here, our aim is to achieve the true intersection curve. To do this, we rotate the Hyperboloid and the plane until inclined plane moves parallel to the XY plane. In this situation, the intersection curve and its projection will be the same. Of course, in this case we will need to use the new equation of Hyperboloid because the Hyperboloid is no longer in standard position, it is rotated and shifted. The same situation is also valid for the intersection of plane and rotational ellipsoid and other quadratic surfaces.

Generally, a Hyperboloid is defined with 9 parameters. These parameters are; 3 coordinates of center (x_o, y_o, z_o), 3 semi-axes (*a,b,c*) and 3 rotational angles (ε, ψ, ω) which represent rotations around *x-,y-* and *z-* axes respectively. These angles control the orientation of the hyperboloid.

2. Intersection of an Hyperboloid and a Plane

The intersection topic is quite popular at an interdisciplinary level. It can be the friends of geometry, geodesy and others. The curves of intersection resulting in this case are not only ellipses but rather all types of conics: ellipses, hyperbolas and parabolas. In text books of mathematics usually only cases are treated, where the planes of intersection are parallel to the coordinate planes. Here the general case is illustrated with intersecting planes which are not necessarily parallel to the coordinate planes. We have developed an algorithm for intersection of a hyperboloid and a plane with a closed form solution. To do this, we rotate the hyperboloid and the plane until inclined plane moves parallel to the XY plane. In this situation, the intersection ellipse and its projection will be the same. This study aims to show how to obtain the center, the semi-axis and orientation of the intersection curve. Here plane equation

$$A_X X + A_Y Y + A_Z Z + A_D = 0 \text{ (Plane equation)} \qquad (2)$$

2.1. The Classification of Intersection

It appears that the classification of hyperbolic conic sections goes back to an 1882 paper by William E. Story [13]:

Story classifies conic sections according to the number and multiplicities of intersections between the conic section and the boundary circle. This results in eight types of conic sections

1. An ellipse is an ellipse contained entirely within the interior of the unit disk.
2. A hyperbola is a conic section that intersects the unit circle at four different points. (Such a conic section may be a portion of an ellipse, parabola, or hyperbola in the Euclidean plane, though which of these three types it is may change under hyperbolic isometries.)
3. A semi-hyperbola is a conic section that intersects the unit circle transversely at two different points. (Again, this may be an ellipse, parabola, or hyperbola in the Euclidean plane.)
4. An elliptic parabola is an ellipse or circle in the disk that intersects the unit circle at one point of tangency.
5. A hyperbolic parabola is a conic section that intersects the unit circle three times, with one being a point of tangency.
6. A semi-circular parabola is a conic section that has the unit circle as one of its osculating circles (i.e. they have a point of third-order contact) and also intersects the unit circle at one additional point.
7. A horocycle (called a "circular parabola" by Story) is an ellipse in the unit disk that has a fourth-order contact with the unit circle. That is, it is an ellipse that has the unit circle as an osculating circle at one of the endpoints of its minor axis.

8. A circle is a circle contained entirely in the interior of the unit disk, and an equidistance conic is an ellipse that is tangent to the unit disk at two points. (Story refers to both of these cases simply as "circles".)

Story mentions that some of these eight types can be further subdivided, and later authors often increased the number of types to 11 or 12. For example, a classification into 12 types can be found on pg. 142 in [14].

Let's assume that X_O, Y_O, Z_O are the coordinates of the center of the intersection curve (ellipses, hyperbolas and parabolas) and a_i, b_i are the major and minor semi-axes of the intersection curve.

We can start with the common solution of two equations (Eq.1-2). If we eliminate the parameter Z, we get the following ellipse equation on the XY plane that is the projection of the intersection curve.

$$A X^2 + B X Y + C Y^2 + D X + E Y + F = 0 \quad (3)$$

These coefficients are calculated from the common solution is obtained from the two equations the hyperboloid and plane equation.

$$\begin{aligned} A &= 1/a^2 + (A_x^2)/(A_z^2 c^2) \\ B &= (2 A_x A_y)/(A_z^2 c^2) \\ C &= 1/b^2 + (A_y^2)/(A_z^2 c^2) \\ D &= (2 A_x A_D)/(A_z^2 c^2) \\ E &= (2 A_y A_D)/(A_z^2 c^2) \\ F &= A_D^2/(A_z^2 c^2) - 1 \end{aligned} \quad (4)$$

When we solve the Eq.3, we get five ellipse parameters. They are:

X_O, Y_O (center of ellipse in XY plane)

a_o, b_o (major and minor semi axis of ellipse in XY plane)

θ (orientation angle between X axis and semi major axis)

$$M_o = \begin{bmatrix} F & D/2 & E/2 \\ D/2 & A & B/2 \\ E/2 & B/2 & C \end{bmatrix} \quad M = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \quad (5)$$

λ_1, λ_2: eigenvalues of M matrices ($\lambda_1 < \lambda_2$)

$$a_o = \sqrt{-\det(M_o)/(\det(M)\lambda_1)}$$

(major semi-axis of intersection ellipse) (6)

$$b_o = \sqrt{-\det(M_o)/(\det(M)\lambda_2)}$$

(minor semi-axis of intersection ellipse) (7)

$$X_O = (B E - 2 C D)/(4 A C - B^2) \quad (8)$$

$$Y_O = (B D - 2 A E)/(4 A C - B^2) \quad (9)$$

(coordinates of intersection ellipse's center)

$$Z_O = -(A_x X_o + A_y Y_o + A_D)/A_z \quad (10)$$

$$\tan 2\theta = \frac{B}{A-C} \quad (11)$$

(orientation angle of projection ellipse)

Now we rotate together the hyperboloid and the plane until inclined plane move parallel to the XY plane. In this situation the intersection ellipse and its projection will be the same.

For this the origin of the XYZ system must be moved to points of P_o (X_O, Y_O, Z_O). We need the transformation parameters.

R_{3x3}-rotation matrix is obtained from the rotational angles

$$R = \begin{bmatrix} \cos\psi\cos\omega & \cos\varepsilon\sin\omega + \sin\varepsilon\sin\psi\cos\omega & \sin\varepsilon\sin\omega - \cos\varepsilon\sin\psi\cos\omega \\ -\cos\psi\sin\omega & \cos\varepsilon\cos\omega - \sin\varepsilon\sin\psi\sin\omega & \sin\varepsilon\cos\omega + \cos\varepsilon\sin\psi\sin\omega \\ \sin\psi & -\sin\varepsilon\cos\psi & \cos\varepsilon\cos\psi \end{bmatrix} \quad (12)$$

This is a 3D transformation equation without scale.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + R \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (13)$$

This transformation equation can be written more simply with **T** Expanded transformation matrix as follows [1, 2, 3].

T_{4x4}- expanded transformation matrix is obtained from the R_{3x3} rotational matrix and the shifted parameters (X_O, Y_O, Z_O)

$$T = \begin{bmatrix} & R_{3x3} & & \begin{matrix} X_0 \\ Y_0 \\ Z_0 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} & R_{3x3}^T & & \begin{matrix} X_{0T} \\ Y_{0T} \\ Z_{0T} \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} = T \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = T^{-1} \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} \quad (15)$$

2.2. Determination of Transformation Parameters

Shifted parameter X_O, Y_O, Z_O are intersection of ellipse's center coordinates that is founded before (Eq.8-10). We must find three rotation angles (ε, Ψ, ω). For this, we take advantage of the nearest plane's point from the origin. The point Q on a plane $A_X.X + A_Y.Y + A_Z.Z + A_D = 0$ that is closest to the origin has the Cartesian coordinates (q_x,q_y,q_z) [11].

Where

$$q_x = \frac{A_x A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \quad q_y = \frac{A_y A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$q_z = \frac{A_z A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \quad (16)$$

And rotation angles (ε, Ψ, ω)

$$\varepsilon = 0 \quad (17)$$

$$\psi = \pi / 2 - \arctan\left(\frac{|q_z|}{\sqrt{q_x^2 + q_y^2}}\right) \quad (18)$$

$$\omega = \pi - \arctan\left(\frac{q_y}{q_x}\right) \quad (19)$$

Of course, in this case we will need to use the new equation of hyperboloid. Because the hyperboloid is no longer in standard position it is rotated and shifted. We have to reverse 3D transformation to the new hyperboloid parameters.

Before we found transformation parameter X_O, Y_O, Z_O, ε, Ψ, ω. These parameters are used transformation from *XYZ* to *xyz*. Let's see how to find the reverse transform parameters (X_{OT}, Y_{OT}, Z_{OT}, ε_T, Ψ_T, ω_T) for the transformation from *xyz* to *XYZ*

To find inverse transformation parameters we can take advantage of the inverse of the T matrix The reverse transformation parameters (X_{OT},Y_{OT}, Z_{OT}, ε_T, Ψ_T, ω_T) are located to T^{-1} inverse matrix. Reverse shifted parameters (X_{OT}, Y_{OT}, Z_{OT}) is located the inverse matrix T^{-1} in column fourth. Reverse rotation angles are calculated from the elements of the matrix R as follows

$$\varepsilon_T = -\text{arc tan}\,(R_{23}/R_{33}) \quad (20)$$

$$\psi_T = \text{arc sin}\,(R_{13}) \quad (21)$$

$$\omega_T = -\text{arc tan}\,(R_{12}/R_{11}) \quad (22)$$

Now we can write a new hyperboloid equation rotated and shifted, to do this, we put (Eq.13). into (Eq.1) standard hyperboloid equation

$$1/a^2[(X-X_{OT}) \cos\Psi_T \cos\omega_T + (Y-Y_{OT})(-\cos\Psi_T) \sin\omega_T + (Z-Z_{OT}) \sin\Psi_T]^2$$
$$+ 1/b^2[(X-X_{OT})(\cos\varepsilon_T \sin\omega_T + \sin\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{OT}) (\cos\varepsilon_T \cos\omega_T - \sin\varepsilon_T \sin\Psi_T \sin\omega_T) - (Z-Z_{OT}) \sin\varepsilon_T) \cos\Psi_T]^2$$
$$- 1/c^2[(X-X_{OT})(\sin\varepsilon_T \sin\omega_T - \cos\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{OT})(\sin\varepsilon_T \cos\omega_T + \cos\varepsilon_T \sin\Psi_T \sin\omega_T) + (Z-Z_{OT}) \cos\varepsilon_T \cos\Psi_T]^2 = \pm 1 \quad (23)$$

In this equation if we put z = 0 we obtain a conical intersection ellipse equation form as follows.

$$1/a^2 [(X-X_{OT}) \cos\Psi_T \cos\omega_T + (Y-Y_{OT})(-\cos\Psi_T) \sin\omega_T - Z_{OT} \sin\Psi_T]^2$$
$$+ 1/b^2 [(X-X_{OT})(\cos\varepsilon_T \sin\omega_T + \sin\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{OT}) (\cos\varepsilon_T \cos\omega_T - \sin\varepsilon_T \sin\Psi_T \sin\omega_T) + Z_{OT} \sin\varepsilon_T) \cos\Psi_T]^2$$
$$- 1/c^2[(X-X_{OT})(\sin\varepsilon_T \sin\omega_T - \cos\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{OT})(\sin\varepsilon_T \cos\omega_T + \cos\varepsilon_T \sin\Psi_T \sin\omega_T) - Z_{OT} \cos\varepsilon_T \cos\Psi_T]^2 = \pm 1 \quad (24)$$

Above conic equation rearranged beloved the intersection ellipse's conic equation obtained.

$$A X^2 + B X Y + C Y^2 + D X + E Y + F = 0 \quad (25)$$

When we solve the above ellipse equation, we get five ellipse parameters of intersection ellipse (X_O, Y_O, a_i, b_i, θ).

As a result we have presented computational results that were realized in MATLAB.

3. Conclusion

In this study, we have developed an algorithm for intersection of a hyperboloid and a plane with a closed form solution. The efficiency of the new approaches is demonstrated through a numerical example. The presented algorithm can be applied easily for spheroid, sphere and also other quadratic surface, such as paraboloid and ellipsoid.

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