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Dirichlet Averages of Wright-Type Hypergeometric Function

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Abstract: In the present paper, the authors approach is based on the use of Dirichlet averages of the generalized Wright-type hyper geometric function introduced by Wright in like the functions of the Mittag-Leffler type, the functions of the Wright type are known to play fundamental roles in various applications of the fractional calculus. This is mainly due to the fact that they are interrelated with the Mittag-Leffler functions through Laplace and Fourier transformations.

Keywords: Dirichlet Averages, Reimann-Liouville Fractional Integral, Wright Type Hyper Geometric Function

1. Introduction

Our translation of real world problems to mathematical expressions relies on calculus, which in turn relies on the differentiation and integration operations of arbitrary order with a sort of misnomer fractional calculus which is also a natural generalization of calculus and its mathematical history is equally long. It plays a significant role in number of fields such as physics, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, elasticity, control theory, engineering mathematics and many others. Fractional calculus like many other mathematical disciplines and ideas has its origin in the quest of researchers for to expand its applicationsto new fields. This freedom of order opens new dimensions and many problems of applied sciences can be tackled in more efficient way by means of fractional calculus.

The purpose of this paper is to increase the accessibility of different dimensions of q-fractional calculus and generalization of basic hypergeometric functions to the real world problems of engineering, science and economics. Present paper reveals a brief history, definition and applications of basic hypergeometric functions and their generalizations in light of different mathematical disciplines.

The Dirichlet averages of a function are a certain kind of integral average with respect to Dirichlet measure. The concept of Dirichlet averages was introduced by Carlson in 1977, based on an integral evaluated by Dirichlet in 1839. It is studied among others by Carlson [1, 2&3], Zu Castell [4], Massopust and Forster [5], Neuman and Van fleet [6] and others. A detailed and comprehensive account of various types of Dirichlet averages has been given by Carlson in his monograph [7]. Most of the important special functions can be represented as Dirichlet averages of the functions $e^x$and x^t, and there are significant advantages in defining them this way instead of by hypergeometric power series. The theory of Dirichlet averages is not restricted to functions of hypergeometric type, because any function that is analytic or even integrable can be averaged with respect to a Dirichlet measure. If the function is twice continuously differentiable, its Dirichlet average satisfies one or more linear second order partial differential equations that are characteristic of the averaging process, and are related to some of the principal differential equations of mathematical physics. In this connection we can refer to the works of Saxena, Kilbas and Sharma etc. Moreover a detailed account on Dirichlet average of a function has been extensively dealt with in the lecture notes of Professor P. K. Banerji [8].

In the present paper we make the use of Riemann-Liouville integrals and Dirichlet integrals which is a multivariate integral and the generalization of a beta integral. Finally, we deduce representations for the Dirichlet averages

$R_k(\beta,\beta'$; x, y) of the Wright type hypergeometric function with the fractional integrals, in particular. Riemann – Liouville integrals. Special cases of the established results associated with Wright type hyper geometric function have also been discussed.

2. Mathematical Preliminaries

Wright type Hypergeometric function: The generalized form of the hypergeometric function has been investigated by Dotsenko [10], Malovichko [11] and one of the special case is considered by Dotsenko [10] as

$$ {}_2R_1^{\omega,\mu}(z) = {}_2R_1(a,b,c,\omega,\mu,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!} $$

And its integral representation expressed as

$$ {}_2R_1^{\omega,\mu}(z) = \frac{\Gamma(c)\mu}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{\mu b-1}(1-t)^{c-b-1}(1-zt^{\omega})^{c-b-1}\text{dt}. \tag{1} $$

Where $\text{Re(c)} > Re(b) > 0.$ This is the analogue of Euler's formula for Gauss's hyper geometric functions [10]. In 2001 Virchenkoetal [9] defined the said Wright type Hypergeometric function by taking $\frac{\omega}{\mu} = \boldsymbol{\tau} > 0$ inabove equation as

$$ {}_2R_1(z) = {}_2R_1(\text{a, b, c,,z}) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!};\ \boldsymbol{\tau} > 0, |z| < 1. \tag{2} $$

If $\boldsymbol{\tau}$=1, then (2) reduces to Gauss's hyper geometric function.

Standard simplex in R^n, n $\geq$ 1: We denote the standard simplex in $R^n, for\ n \geq 1$ by

$$ \text{E} = E_n = (u_1, u_2, \dots u_n);\ u_1 \geq 0, u_2 \geq 0, \dots, u_n \geq 0 \text{ and } u_1 + u_2 + u_3 + u_n \leqq 1\}. $$

Dirichlet Measures: Let b ϵc^k>; k $\geq$ 2and let E = E_{k-1} be the standard simplex in R^{k-1}. The complex measure μ_b is defined by [1]

$$ d_{\mu_b}(\text{u}) = \frac{1}{B(b)}u_1^{\,b_1-1}u_2^{\,b_2-1}u_3^{\,b_3-1}\dots u_k^{\,b_{k-1}-1} \times $$

$$ (1-u_1, 1-u_2, \dots 1-u_{k-1})^{b_k-1}d_{u_1}d_{u_2}d_{u_3}\dots d_{u_{k-1}}. $$

Here $\text{B(b)} = \text{B}(b_1, b_2, \dots, b_k) = \frac{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_k)}{\Gamma(b_1+b_2+\cdots . b_k)}$

Dirichlet Averages: Dirichlet averages are discussed in the book by Carlson [1] and related to univariate and Multivariate B-splines in [3]. Dirichlet averages have produced deep and interesting connections to special functions. In this section, we extend the notion of the Dirichlet average to the infinite dimensional setting and show that under mild conditions on the weights, the results important for our interests do also hold on Δ^∞. In particular, we show that using a geometric interpretation, the Reimann-Liouville fractional derivative and integral can be applied to Dirichlet averages.

Let Ωbe a convex set in C and let z = $(z_1, z_2, \dots z_n)\ \epsilon\Omega^{\boldsymbol{n}}$, n≥2, and let f be a measurable function on Ω. We define

$\text{F (b; z)} = \int_{E_{n-1}} f(uoz)\, d_{\mu_b}(u)$, where d_{μ_b}(u) is a Dirichlet Measure.

$\text{B (b)} = \text{B}(b_1, b, \dots, b_n) = \frac{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_n)}{\Gamma(b_1+b_2+\cdots+b_n)}$, R($b_j$) > 0, j = 1,2,3…,n

And

$$ uoz = \sum_{j=1}^{n-1}u_j z_j + (1-u_1 \ \dots -u_{n-1})\, z_n. $$

For n = 1, $f(b;z) = f(z)$,
for n = 2, we have

$$ d_{\mu_{\beta,\beta'}}(\text{u}) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}u^{\beta-1}(1-u)^{\beta'-1}\text{d(u)}. $$

Carlson [10] investigated the average as follows:

Dirichlet Averages for the power function: $f(z) = z^k$, kϵR,

$$ R_k(\text{b;z}) = \int_{E_{n-1}} (uoz)^k\, d_{\mu_b}(u), \ (\text{k}\epsilon\text{R}). $$

Also for n = 2, Carlson proved that

$$ R_k(\beta,\beta';x,\text{y}) = \frac{1}{B(\beta,\beta')}\int_0^1[ux+(1-u)y]^k\, u^{\beta-1}(1-u)^{\beta'-1}\text{d(u)}, \tag{3} $$

Where $\beta,\beta'\epsilon$C, min [R (β), R(β')] > 0, $x, y\epsilon\ R$.

This section is devoted to the study of the Dirichlet averages of the Wright-type hypergeometric function (2) in the form

$$ {}_2M_1^{\mu,\ \omega}\left[\begin{pmatrix} a, b \\ c \end{pmatrix}\middle|(\beta,\beta';\text{x,y})\right] = \int_{E_1} {}_2R_1^{\mu,\omega}(uoz)\, d_{\mu_{\beta\beta'}}(\text{u}) \tag{4} $$

Where R(β) > 0, R(β') > 0; $x, y\ \epsilon\ R$ and $\beta,\beta'\epsilon$ C.

Fractional Calculus and its Elements:

The concept of fractional calculus is not new. It is believed to have stemmed from a question raised by L"Hospital on September 30th, 1695, in a letter to Leibniz, about $\frac{d^n y}{dx^n}$, Leibniz"s notation for the nth derivative of the linear function f(x)= x. L"Hospital curiously asked what the result would be if n= $\frac{1}{2}$ Leibniz responded prophetically that it would be an apparent paradox from which one day useful consequences would be drawn.

Following this unprecedented discussion, the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development. This included Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville. Over the years, many mathematicians, using their own notation and approach, have found various definitions that fit the idea of a non-integer order integral or derivative. One version that has been popularized in the world of fractional calculus is the Riemann-Liouville definition. It is interesting to note that the Riemann-Liouville definition of a fractional derivative gives the same result as that obtained by Lacroix.

In this section we present a brief sketch of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied, it involves the Riemann-Liouville fractional operators, weyl operators and Saigo"s operators etc. There are more than one version of the fractional integral operator exist. The fractional integral can be defined as follows:

Riemann-Liouville fractional integrals: As defined in [6]

$$(I_{a+}^{\alpha} f)\, x = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\, dt, \ (x > a, a\epsilon R). \qquad (5)$$

Thus, in general the Riemann-Liouville fractional integrals of arbitrary order for a function f(t), is a natural consequence of the well-known formula (Cauchy-Dirichlets?) that reduces the calculation of the n-fold primitive of a function f (t) to a single integral of convolution type.

3. Main Results

Representation of R_k and ${}_2M_1^{\mu,\omega}$ in terms of Reimann-Liouville Fractional Integrals.

In this section we deduce representations for the Dirichlet averages $R_k(\beta, \beta', x, y)$and ${}_2M_1^{\mu,\omega}(\beta,\beta'; x, y)$ with fractional integral operators.

Theorem: Let $\beta, \beta' \epsilon$ complex numbers, $R(\beta) > 0$, $R(\beta') > 0$, and x,y be real numbers such that $x > y$ and $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$,and ${}_2M_1^{\mu,\omega}$ and I_{a+}^{α} be given by (4) and (5) respectively. Then the Dirichlet average of the generalized Fox- wright functions is given by

$$ {}_2M_1^{\mu,\ \omega}\left[\left(\begin{matrix}a,b\\c\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] = \frac{\Gamma(\beta+\beta\prime)}{\Gamma(\beta)(x-y)^{\beta+\beta\prime-1}}\left[(I_{0+}^{\alpha}\ {}_2R_1(a,b,c,\omega,\mu,z)\right].$$

Where $\beta, \beta' \epsilon$ C, $R(\beta) > 0$, $R(\beta') > 0$, x,y ϵ R and $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{q} A_j \geq 0$ (equality only holds for appropriately bounded z).

Proof: According to equations (1) and (2) we have,

$$ {}_2M_1^{\mu,\ \omega}\left[\left(\begin{matrix}a,b\\c\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] = \frac{1}{B(\beta,\beta\prime)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)n!}\int_0^1[y+u(x-y)]^n u^{\beta-1}(1-u)^{\beta'-1}d(u).$$

Or

$$ {}_2M_1^{\mu,\ \omega}\left[\left(\begin{matrix}a,b\\c\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] = \frac{\Gamma(\beta+\beta\prime)}{\Gamma(\beta\prime)\Gamma(\beta)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)n!}\int_0^1[y+u(x-y)]^n u^{\beta-1}(1-u)^{\beta'-1}d(u).$$

Putting $u(x-y) = t$ in above equation, we get

$$ {}_2M_1^{\mu,\ \omega}\left[\left(\begin{matrix}a,b\\c\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta')\Gamma(\beta)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)n!}\int_0^{x-y}[y+t]^n\left\{\frac{t}{x-y}\right\}^{\beta-1}\left(1-\frac{t}{x-y}\right)^{\beta'-1}\frac{dt}{x-y}$$

$$= \frac{(x-y)^{1-\beta-\beta\prime}}{B(\beta,\beta\prime)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)n!}\int_0^{x-y}[y+t]^n\{t\}^{\beta-1}(x-y-t)^{\beta'-1}dt$$

$$= \frac{(x-y)^{1-\beta-\beta'}}{B(\beta,\beta')}\int_0^{x-y} t^{\beta-1}\ {}_2R_1(a,b,c,\omega,\mu,|y+t)\ (x-y-t)^{\beta'-1}dt$$

$$= \frac{(x-y)^{1-\beta-\beta'}}{B(\beta,\beta')}\int_0^{x-y} t^{\beta-1}\ {}_2R_1(a,b,c,\omega,\mu,|y+t)\ (x-y-t)^{\beta'-1}dt$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left[(I_{0+}^{\alpha}\ {}_2R_1(a,b,c,\omega,\mu,z)\right]$$

This proves the theorem.

4. Conclusion

Here we concluded with that that Dirichlets average of a function denotes certain kind of *integral average* with respect to a Dirichlets measure. Most of the important special functions can be represented as Dirichlets averages of the functions e^x and x^t, and there are significant advantages in defining them this way instead of by hypergeometric power series. The theory of Dirichlets averages is not restricted to functions of hypergeometric type, because any function that is analytic or even integrable can be averaged with respect to a Dirichlets measure. If the function is twice continuously differentiable, its Dirichlets average satisfies one or more linear second order partial differential equations that are characteristic of the average process, and are related to some of principal differential equations of mathematical physics. In this connection one can refer to the works of Saxena, Kilbas and Sharma etc. Moreover for the detailed account on Dirichlets average of a functionone should go through lectures of Professor P. K. Banerji.

Finally we conclude with the remark that the results and the operators proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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