International Journal of Discrete Mathematics
2017; 2(1): 1-5
http://www.sciencepublishinggroup.com/j/dmath
doi: 10.11648/j.dmath.20170201.11

The Classical Laplace Transform and Its q-Image of the Most Generalized Hypergeometric and Mittag-Leffler Functions

D. K. Jain[1], Altaf Ahmad[2], Renu Jain[2], Farooq Ahmad[3]

[1]Madhav Institute of Technology and Science, Gwalior (M.P.), India

[2]School of Mathematics and Allied Sciences Jiwaji University, Gwalior (M.P.), India

[3]Department of Mathematics, Govt. Degree College, Kupwara (J&K), India

Email address:

jain_dkj@yahoo.co.in (D. K. Jain), altaf.u786@gmail.com (A. Ahmad), renujain3@rediffmail.com (R. Jain), sheikhfarooq85@gmail.com (F. Ahmad)

To cite this article:

D. K. Jain, Altaf Ahmad, Renu Jain, Farooq Ahmad. The Classical Laplace Transform and its q-Image of the Most Generalized Hypergeometric and Mittag-Leffler Functions. *International Journal of Discrete Mathematics.* Vol. 2, No. 1, 2017, pp. 1-5. doi: 10.11648/j.dmath.20170201.11

Received: December 23, 2016; **Accepted**: January 16, 2017; **Published**: February 9, 2017

Abstract: The q-Calculus has served as a bridge between mathematics and physics, particularly in case of quantum physics. The q-generalizations of mathematical concepts like Laplace and Fourier transforms, Hypergeometric functions etc. can be advantageously used in solution of various problems arising in the field of physical and engineering sciences. The present paper deals with some of the important results of q-Laplace transform of Fox-Wright and Mittag-Leffler functions in terms of well-known Fox's H-function. Some special cases have also been discussed.

Keywords: Classical Laplace Transform, q-Image of Laplace Transform, ML-Function, Fox-Wright Function

1. Introduction

Our translation of real world problems to mathematical expressions relies on calculus which has been generalized in several directions. A natural generalization of calculus, called fractional calculus was developed during eighteenth century which involved the differentiation and integration operations of arbitrary order, which is a sort of misnomer. In the beginning it did not develop sufficiently due to lack of applications. Over the years various applications of the concept were explored and the efforts were so rewarding that the subject itself has been categorized as a significant branch of applicable mathematics. It plays a significant role in number of fields such as physics, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, elasticity, control theory, engineering mathematics and many others.

In order to stimulate more interest in the subject, many research workers engaged their focus on another dimension of calculus which sometimes called calculus without limits or popularly q-calculus. The q-calculus was initiated in twenties of the last century. Kac and Cheung's book [13] entitled "Quantum Calculus" provides the basics of such type of calculus. The fractional q-calculus is the q-extension of the ordinary fractional calculus. The investigations of q-integrals and q-derivatives of arbitrary order have gained importance due to its various applications in the areas like ordinary fractional calculus, solutions of the q-difference (differential) and q-integral equations, q-transform analysis etc.

Hypergeometric functions evolved as natural unification of a host of functions discussed by analysts from the seventeenth century to the present day. Functions of this type may also be generalized using the concept of basic number. Over the last thirty years, a great resurgence of interest in q-functions has been witnessed in view of their application to number theory and other areas of mathematics and physics.

The Mittag–Leffler function and Fox-Wright functions are generalizations of Hypergeometric functions which appear as solution of well-known fractional differential and integral equations representing some physical and physiological phenomena like diffusion, transport theory, probability, elasticity and control theory.

The purpose of this paper is to increase the accessibility of different dimensions of q-fractional calculus and generalization of basic hypergeometric functions to the real world problems of engineering and science through various integral transforms including Laplace and Fourier transforms and their q-images.

The classical Laplace, Fourier and Mellin transforms have been widely used in mathematical physics and applied mathematics. The classical theory of the Laplace transform is well known Sneddon [1] and its generalization was considered by many authors [2, 3, 4, 8 and 9]. Various existence conditions and the detailed study about the range and invertibility was studied by Rooney [7]. The Laplace transform and Mellin transform are widely used together to solve the fractional kinetic equations and thermonuclear equations [5].

2. Mathematical Preliminaries

Classical Laplace transform: The Laplace transform is very useful in analysis and design for systems that are linear and time-invariant (LTI). Beginning in about 1910, transform techniques were applied to signal processing at Bell Labs for signal filtering and telephone long-lines communication by H. Bode and others. Transform theory subsequently provided the backbone of Classical Control Theory as practiced during the World Wars and upto about 1960, when State Variable techniques began to be used for controls design. Pierre Simon Laplace was a French mathematician who lived 1749-1827, during the age of enlightenment characterized by the French Revolution, Rousseau, Voltaire and Napoleon Bonaparte. Let $f(t)$ be a function piecewise continuous on $[0,A]$ (for every $A>0$) and have an exponential order at infinity with$f(t) \leq Me^{at}$. Then, the Laplace transform L(f) is defined for s > a, that is {s >a}⊂Domain(L(f)). The Laplace transform of f(t) is defined by

$$L[f(t)]= F(s) = \int_0^{\infty} e^{-st} f(t)\,dt . \quad (1)$$

The Laplace transform is said to exist if the integral (1) is convergent for some values of s.

Classical Fourier Transform: Fourier analysis is named after Jean Baptiste Joseph Fourier (1768to1830), a French mathematician and physicist. Joseph Fourier, while studying the propagation of heat in the early 1800's, introduced the idea of a harmonic series that can describe any periodic motion regardless of its complexity. Later, the spelling of Fourier analysis gave place to Fourier transform (FT) and many methods derived from FT are proposed by researchers. In general, FT is a mathematical process that relates the measured signal to its frequency content Heideman et.al.[12]. The Fourier series is described in theory and problems of advanced calculus as follows:

If f(x) be a function defined on $(-\infty,\infty)$ uniformly continuous in finite interval such that $\int_0^{\infty}\|f(x)\|d(x)$ converges, then the Fourier transform of f(x) is defined by

$F(f(x))= \bar{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, d(x)$, where e^{isx} is said to be kernel of the Fourier transform.

q-image of Laplace transform: Hahn[6] defined the q-image of classical Laplace transform as

$$L_q f(s) = \int_0^{\infty} e_q^{-sx} f(x)d(x), Re(s)>0. \quad (2)$$

Where e_q^{-sx} is defined by $e_q^{-sx} = \frac{1}{(1+s(1-q)x)_q^{\infty}}$.

The Laplace transform of the power function is defined as

$$L(t^{\mu})=\frac{\Gamma(\mu+1)}{s^{\mu+1}} \quad (3)$$

The q-Laplace transform of the power function is defined as in [10&11]

$$L_q(t^{\mu})=\frac{\Gamma_q(\mu+1)(1-q)^{\mu}}{s^{\mu+1}} \quad (4)$$

Also, $(1-q)^{\alpha-1}\,\Gamma_q(\alpha) = (q;q)_{\alpha-1}$

Fox-Wright generalized hypergeometric Function:

The Fox-Wright (Psi) function is defined as follows [14].

$$ {}_p\psi_q\left(\begin{matrix}(a_1,A_1),(a_2,A_2),...(a_p,A_p)\\(a_1,A_1),(a_2,A_2),...(a_p,A_p)\,|\,z\end{matrix}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)..\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\frac{z^n}{n!} \quad (5)$$

Where $a\,i$, $b\,j$ ϵ C >0; $A\,i > 0$, $B\,j > 0$; $1+\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} Ai \geq 0$; a ϵR, for suitably bounded value of |z|.

The basic analogue of Fox-Wright hypergeometric function denoted ${}_p\psi_q(z;q)$ for z ϵ C is defined in series form as [18]

${}_p\Psi_q(z;q)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_q(a_i+kA_i)}{\prod_{j=1}^{q}\Gamma_q(b_j+kB_j)}\frac{z^k}{(q;q)_k}$, where |q|<1.

Mittag-Leffler Function:

The Mittag-Leffler function is named after a Swedish mathematician who defined and studied it. The function is a direct generalization of the exponential function, e^x and it plays a major role in fractional calculus. The one, two and three-parameter representations of the Mittag-Leffler function can be defined in terms of a power series as follows [15, 16, 17].

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\alpha n)}, \text{ for } \alpha \in C, R(\alpha)>0. \quad (6)$$

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\beta+\alpha n)}\text{, for } \alpha,\beta \in C, R(\alpha)>0. \quad (7)$$

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^{n}}{\Gamma(\beta+\alpha n)}\frac{1}{n!}\text{, for } \alpha,\beta,\gamma \in C, R(\alpha)>0. \quad (8)$$

where, $(\gamma)_n=\gamma(\gamma+1)(\gamma+2)(\gamma+3)\ldots(\gamma+n-1)$ and $(\gamma)_0=1$.

For $\alpha,\beta,\gamma \epsilon \boldsymbol{C}, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0$ and $|q|<1$ the function $E^{\gamma}_{\alpha,\beta}(z;q)$ is defined as[19]:

$E^{\gamma}_{\alpha,\beta}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n}{(q;q)_n}\frac{z^n}{\Gamma_q(\alpha n+\beta)}$, where $\Gamma_q(\lambda)$ is the gamma function.

3. Main Results

(A) In this section of paper, the authors have derived the classical Laplace transform of Fox-Wright and Mittag-Leffler functions in terms of Fox's H–function.

Theorem 1: The classical Laplace transform of Fox-Wright function in terms of Fox's H–function is given by

$$L\left\{ p\psi q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z \right)\right\} =$$

$$\frac{1}{s}H^{1,p}_{1,q}\left[\begin{matrix} (1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)\ldots(1-a_p,-A_p) \\ (1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)\ldots(1-b_q,-B_q) \end{matrix} \middle| S \right]$$

Proof: For $R(\alpha)>0$, the classical Laplace transform of Fox-Wright in terms of Fox's H–function is given by

$$L\left\{ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z \right)\right\} =$$

$$L\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)}\frac{z^n}{n!}\right\} \quad (9)$$

From equation (9) we have,

$$L\left\{ p\psi q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z \right)\right\} =$$

$$\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)}\right\} L\left(\frac{z^n}{n!}\right)$$

Or by using equation (3) we get,

$$L\left\{ p\psi q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z \right)\right\} =$$

$$\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)}\frac{1}{\Gamma(n+1)}\right\}\frac{\Gamma(n+1)}{s^{n+1}}$$

Which implies that,

$$L\left\{ p\psi q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z \right)\right\} = \frac{1}{s}$$

$$\sum_{n=0}^{\infty}\frac{\Gamma(1-(1-a_1)+nA_1)\Gamma(1-(1-a_2)+nA_2)\ldots\Gamma(1-(1-a_p)+nA_p)}{\Gamma(1-(1-b_1)+nB_1)\Gamma(1-(1-b_2)+nB_2)\ldots\Gamma(1-(1-b_q)+nB_q)}s^{-n}$$

$$L\left\{ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z \right)\right\} =$$

$$\frac{1}{s}H^{1,p}_{1,q}\left[\begin{matrix} (1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)\ldots(1-a_p,-A_p) \\ (1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)\ldots(1-b_q,-B_q) \end{matrix} \middle| S \right].$$

This is the proof of theorem.

Theorem 2: The classical Laplace transform of ML-Function in terms of Fox's H–function is given by

$$L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{s}\frac{1}{\Gamma(\gamma)}H^{1,1}_{1,1}\left[\begin{matrix} (1-\gamma,-1) \\ (1-\beta,-\alpha) \end{matrix} \middle| S \right]$$

Proof: For $R(\alpha)>0$, the classical Laplace transform of ML-Function in terms of Fox's H–function is given by

$$L(E^{\gamma}_{\alpha,\beta}(z)) = L\left\{\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)}\frac{1}{n!}\right\} \quad (10)$$

Since, $(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$

Therefore, from equation(10) we have,

$$L(E^{\gamma}_{\alpha,\beta}(z)) = L\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)z^n}{\Gamma(\beta+\alpha n)\Gamma(\gamma)}\frac{1}{n!}\right\}$$

Or $L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\beta+\alpha n)}\right\}\frac{1}{\Gamma(n+1)}L(z^n)$

Or by using equation(3) we get,

$$L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\beta+\alpha n)}\right\}\frac{1}{\Gamma(n+1)}\frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{Or } L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{s\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\beta+\alpha n)}\right\} s^{-n}$$

$$\text{Or } L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{s\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma 1-(1-\gamma)+n}{\Gamma(1-(1-\beta)+\alpha n}s^{-n}\right\}$$

$$\text{Or } L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{s\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma 1-(1-\gamma)-(-n)}{\Gamma(1-(1-\beta)+\alpha n}s^{-n}\right\}$$

Which implies,

$$\text{Or } L(E^{\gamma}_{\alpha,\beta}(z)) = \frac{1}{s\Gamma(\gamma)} H^{1,1}_{1,1}\left[\begin{matrix}(1-\gamma,-1)\\(1-\beta,-\alpha)\end{matrix}\middle| S\right].$$

This is the proof of the theorem.

Observations:

(1.1): If $\gamma=1$ then from above theorem

$$L(E_{\alpha,\beta}(z) = \frac{1}{s} H^{1,1}_{1,1}\left[\begin{matrix}(0,-1)\\(1-\beta,-\alpha)\end{matrix}\middle| S\right].$$

(1.2): If $\gamma=1$ and $\beta=1$ then from above theorem

$$L(E_{\alpha}(z) = \frac{1}{s} H^{1,1}_{1,1}\left[\begin{matrix}(0,-1)\\(0,-\alpha)\end{matrix}\middle| S\right].$$

(B) In this section of paper, the authors have derived the q-image Laplace transform of basic analogue of Fox-Wright and Mittag-Leffler functions in terms of Fox's q-analogue of H–function which is given by

Theorem3: The q-Laplace transform of q-analogue of Fox-Wright Function in terms of q-analogue of H–function is given by

$$L_q\left\{ \text{p}\psi\text{q}\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\ldots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\ldots(b_q,B_q)\end{matrix}\middle| z;q\right)\right\} =$$

$$\frac{1}{s}H^{1,p}_{1,q}\left[\begin{matrix}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)\ldots(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)\ldots(1-b_q,-B_q)\end{matrix}\middle| S;q\right]$$

Proof: For $q>0, R(\alpha)>0$, the q-image of Laplace transform of q-type of Fox-Wright Function in terms of basic analogue of H–function is given by

$$L_q\left\{ \text{p}\psi\text{q}\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\ldots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\ldots(b_q,B_q)\end{matrix}\middle| z;q\right)\right\} =$$

$$L_q\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(a_1+nA_1)\Gamma_q(a_2+nA_2)\ldots\Gamma_q(a_p+nA_p)}{\Gamma_q(b_1+nB_1)\Gamma_q(b_2+nB_2)\ldots\Gamma_q(b_q+nB_q)}\frac{z^n}{(q;q)_n}\right\} \quad (11)$$

From equation (11) we have,

$$Lq\left\{ {}_p\psi_q\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\ldots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\ldots(b_q,B_q)\end{matrix}\middle| z;q\right)\right\} =$$

$$\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(a_1+nA_1)\Gamma_q(a_2+nA_2)\ldots\Gamma_q(a_p+nA_p)}{\Gamma_q(b_1+nB_1)\Gamma_q(b_2+nB_2)\ldots\Gamma_q(b_q+nB_q)}\right\} L_q\left(\frac{z^n}{(q;q)_n}\right)$$

Or by using equation (4) we get,

$$Lq\left\{ {}_p\psi_q\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\ldots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\ldots(b_q,B_q)\end{matrix}\middle| z;q\right)\right\} =$$

$$\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(a_1+nA_1)\Gamma_q(a_2+nA_2)\ldots\Gamma_q(a_p+nA_p)}{\Gamma_q(b_1+nB_1)\Gamma_q(b_2+nB_2)\ldots\Gamma_q(b_q+nB_q)}\right\}\frac{1}{\Gamma_q(n+1)}\frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{1}{s}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(1-(1-a_1)+nA_1)\Gamma_q(1-(1-a_2)+nA_2)\ldots\Gamma_q(1-(1-a_p)+nA_p)}{\Gamma_q(1-(1-b_1)+nB_1)\Gamma_q(1-(1-b_2)+nB_2)\ldots\Gamma_q(1-(1-b_q)+nB_q)}\right\}s^{-n}$$

Which implies that,

$$Lq\left\{ {}_p\psi_q\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\ldots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\ldots(b_q,B_q)\end{matrix}\middle| z;q\right)\right\} =$$

$$\frac{1}{s}H^{1,p}_{1,q}\left[\begin{matrix}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)\ldots(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)\ldots(1-b_q,-B_q)\end{matrix}\middle| S;q\right]$$

This is the proof of theorem.

Theorem 4: The q-Laplace transform of q-analogue of ML-Function in terms of q-analogue of H–function is given by

$$L_q(E^{\gamma}_{\alpha,\beta}(z:q) = \frac{1}{s\Gamma_q(s)} H^{1,1}_{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}\middle| S;q\right]$$

Proof: For $q>0, R(\alpha)>$, the q-image of Laplace transform of q-type of ML-Function in terms of basic analogue of H–function is given by

$$L_q(E^{\gamma}_{\alpha,\beta}(z:q) = L_q\left\{\sum_{n=0}^{\infty}\frac{(\gamma;q)_n z^n}{\Gamma_q(\beta+\alpha n)}\frac{1}{(q;q)_n}\right\}$$

$$\text{Or } L_q(E^{\gamma}_{\alpha,\beta}(z:q) = \frac{1}{\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(\gamma+n)}{\Gamma_q(\beta+\alpha n)}\right\}\frac{1}{(q;q)_n}L_q(z^n)$$

By making use of equation (4) we get,

$$\mathrm{Lq}\,(E^{\gamma}_{\alpha,\beta}(z:q) = \frac{1}{\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(\gamma+n)}{\Gamma_q(\beta+\alpha n)}\right\}\frac{1}{(q;q)_n}\frac{(q;q)_n}{s^{n+1}}$$

$$\text{Or } \mathrm{L_q}\,(E^{\gamma}_{\alpha,\beta}(z:q) = \frac{1}{s\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(\gamma+n)}{\Gamma_q(\beta+\alpha n)}\right\}\frac{1}{s^n}$$

Or $\mathrm{Lq}\,(E^{\gamma}_{\alpha,\beta}(z:q)$

$$= \frac{1}{s\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(1-(1-\gamma)+n)}{\Gamma_q(1-(1-\beta)+\alpha n)}\right\}\frac{1}{s^n}$$

$$\text{Or } \mathrm{Lq}\,(E^{\gamma}_{\alpha,\beta}(z:q) = \frac{1}{s\Gamma_q(\gamma)} H^{1,1}_{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}\middle| S;q\right]$$

This completes the proof.

Observations:

(1.3): If $\gamma=1$ then from above theorem

$$L_q(E_{\alpha,\beta}(z)=\frac{1}{s}H^{1,1}_{1,1}\left[\begin{matrix}(0,1)\\(1-\beta,\alpha)\end{matrix}\middle| S;q\right],$$

(1.4): if $\gamma=1$ & $\beta=1$ then from above theorem

$$L_q(E_{\alpha}(z)=\frac{1}{s}H^{1,1}_{1,1}\left[\begin{matrix}(0,1)\\(0,\alpha)\end{matrix}\middle| S;q\right].$$

4. Special Cases

Taking q=1, we get following as special case of theorem (4)

$$L(E^{\gamma}_{\alpha,\beta}(z)=\frac{1}{s\Gamma(\gamma)}H^{1,1}_{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}\middle| S\right]$$

if $\gamma=1$ then from above theorem

$$L(E_{\alpha,\beta}(z)=\frac{1}{s}H^{1,1}_{1,1}\left[\begin{matrix}(0,1)\\(1-\beta,\alpha)\end{matrix}\middle| S\right]$$

If $\gamma=1$ and $\beta=1$ then from above theorem

$$L(E_{\alpha}(z)=\frac{1}{s}H^{1,1}_{1,1}\left[\begin{matrix}(0,1)\\(0,\alpha)\end{matrix}\middle| S\right]$$

5. Conclusion

The results proved in this paper give some contributions to the theory of the q-series, especially q-analogue of generalized hypergeometric function and Mittag-Leffler Function and may find applications to solutions of certain q-difference, q-integral and q-differential equations with the help of q-images of transforms like Laplace and Fourier transforms. The results proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

References

[1] Sneddon, I. N. (1995). Fourier transforms. Courier Corporation.

[2] Zemanian, A. H. (1968). Generalized integral transform, Pure Appl. Math. 18, inter Science Publ. John Wiley and Sons, New York.

[3] Rao, G. L. N. (1974). The Generalized Laplace transform of generalized function, Ranchi Univ, Math. J.5.

[4] Saxena, R. K. (1960). Some theorems on generalized Laplace transform I, Proc. Nat. Inst. Sci. India, part A, 26,400-413.

[5] Mathai, A. M. (2007). A Versatile integral, Preprint series, No.12, Centre for Mathematical Sciences, Pala, Kerala, India.

[6] Hahn, W. (1949). Beitrage Zur Theorie der Heineshan Reihen. Die 24 integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation. Mathematische Nachrichten, 2(6), 340-379.

[7] Rooney, P. G. (1982\83). On integral transformation with G-function kernels, proc. Royal. Soc. Edinburgh Sect. A. 93.

[8] Saxena, R. K. (1961). Some theorems on generalized Laplace transform II, Riv. Mat. Univ. Parma(2)2, 287-299.

[9] Saxena, R. K. (1966). Some theorems on generalize Laplace transform, Proc. Cambridge Philos. Soc; 62, 467-471.

[10] Purohit, S. D., Yadav R. K. and Vyas, V. K. On applications of q-Laplace transforms to a basic analogue of the I-function. (Communicated).

[11] Purohit S. D., and Yadav, R. K. On q-Laplace transforms of certain q-hypergeometric polynomials , Proc. Nat. Acad. Sci. India(2006) 235-242-III.

[12] Heideman, M. T., Johnson, D. H., & Burrus, C. S. (1985). Gauss and the history of the fast Fourier transform. Archive for history of exact sciences, 34(3), 265-277.

[13] Kac, V., & Cheung, P. (2001). Quantum calculus. Springer Science & Business Media.

[14] Wright, E. M (1935), The asymptotic expansion of generalized hypergeometric function. J. London Math. So, 10, 286-293.

[15] Mittag-Leffler function, M. G.: 1903 Sur la nouvelle function $\boldsymbol{E}_{\propto}(\boldsymbol{x})$, Comptes Rendus Acad. Sci, Paris (Ser.II)137, 554-558.

[16] Metzler, R and Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach, phys. Rep. 339(2000), 1-77.

[17] Prabhakar, T. R (1971), A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. Vol. 19, 7-15.

[18] Sheikh, F. A., & Jain, D. K. Basic Hypergeometric Functions and q-Calculus. Corona publications (2015).

[19] Sharma, S. K., & Jain, R. On Some Properties Of Generalized q-Mittag-Leffler Function. Mathematica Aeterna, Vol. 4, 2014, no.6, 613–619.